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A Monte Carlo study on the spin dynamics of the 2D $\pm J$ Ising spin glass model

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Abstract. We examine the dynamical properties of the 2D $\pm J$ Edwards–Anderson Ising spin glass in the anomalous phase which corresponds to the Griffiths phase in the diluted ferromagnetic system. By using Monte Carlo simulation we find that the autocorrelation function can be described as a function of a power of scaled time. This new scaling includes both the stretched exponential law for the intermediate stage and the logarithmic law for the early stage of relaxation. The relaxation time obtained from the scaling is found to diverge exponentially with exponent proportional to T^{-2}.

1. Introduction

The dynamical properties of spin glasses (SG) forms one of the most important and interesting subjects to study for understanding the slow dynamics widely seen in various complex systems. In fact a number of phenomena exhibiting anomalously slow relaxation, such as those obeying power-law, enhanced power-law and stretched exponential-law behaviour, have been found experimentally. There is, however, no satisfactory explanation for them on the basis of microscopic theory.

For spin glasses it has been suggested [1] that there exists a kind of Griffiths phase [2]. The essential idea of the phenomenology is based on a cluster theory, namely, that unfrustrating spin clusters in an SG system play the role of bonding clusters in the corresponding dilute ferromagnetic system and contribute to the construction of an anomalous phase below the transition temperature of the pure ferromagnetic system, T_c^{pure}. These clusters are considered to make the dynamics extremely slow. The cluster theory predicts that the autocorrelation function obeys the enhanced power law in the Griffiths phase [1], and for the dilute ferromagnet, the prediction is consistent with some numerical experiments [3–5]. On the other hand, Ogielski [6] observed by his enormous simulation for the 3D $\pm J$ SG system that the autocorrelation function follows the stretched exponential decay between T_c^{pure} and the SG transition temperature, T_g. Although his result suggests the existence of the Griffiths phase, the relaxational dynamics contradicts that of the cluster theory.

To investigate the slow dynamics of the SG system more precisely, we examine, in this paper, the autocorrelation function of the 2D $\pm J$ Ising SG system. If the anomalous phase mentioned above is associated with the Griffiths phase, the same phase is also expected to exist in the 2D SG system even though the SG transition does not occur, i.e. $T_g = 0$, whereas $T_c^{\text{pure}} \neq 0$. Actually some evidence for its existence has been reported in [7], where both the spatial spin correlation and the autocorrelation

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were examined by Monte Carlo simulation, and which is partly a preliminary work of this paper.

As the main topic of this paper, we propose a new scaling hypothesis to describe the behaviour of the autocorrelation function. This scaling is found to work well at the early and intermediate stages of the relaxation, where our data are compatible with the stretched exponential law. In the following section we describe the spin model briefly, and our new scaling result is shown in section 3. The comparison to other scaling predictions is argued in section 4 and section 5 is devoted to discussion. Some numerical details of the scaling are explained separately in the appendix.

2. Monte Carlo method

We study a standard model of spin glasses which is described by the Hamiltonian

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j \tag{1}$$

where $\{S_i\}$ are the Ising spins on a square lattice with skew boundary conditions and the nearest-neighbour interactions $\{J_{ij}\}$ are given as independent random variables obeying the symmetric $\pm J$ distribution.

The Monte Carlo simulation was performed on special-purpose hardware, m-TIS2, attached to an NEC PC9801 microcomputer. This hardware is specialized to simulate Ising systems by single-spin-flip Monte Carlo dynamics [8]. We adopted the so-called sequential flip for the update order and the heat-bath method for the flipping rate.

As mentioned in the introduction, the physical quantity we concentrated on is the spin autocorrelation function, defined by

$$q(t) = \frac{1}{N}\sum_{i}^{N} \langle S_i(0) S_i(t) \rangle \tag{2}$$

where $\langle \ldots \rangle$ denotes a statistical mechanical average. We observed $q(t)$ for a wide range of temperatures ($0.8 \leqslant T \leqslant 5.0$ in units of J) around the transition temperature of the pure Ising system, $T_c^{\text{pure}} \simeq 2.27$. In figure 1 we show the raw data for a system of linear dimension $L = 400$. The Monte Carlo steps used are 20 000–2 000 000, depending on temperature, so as to make the standard deviation less than 10^{-3} (typically 2×10^{-4}). We also observed the same quantity with $L = 50$ and 100, and confirmed that there is no considerable finite-size effect in the data shown in the figure.

3. Results

3.1. *Scaling of* $q(t)$

We tried to scale $q(t)$ to the form

$$q(t) = Q[(t/\tau)^A] \tag{3}$$

where $Q(z)$ is a universal scaling function, and A and τ are parameters to be determined for each T. Here we illustrate schematically how we can scale our data to (3). The three right-hand curves in figure 2(*a*) represent our raw data on a logarithmic scale of t.

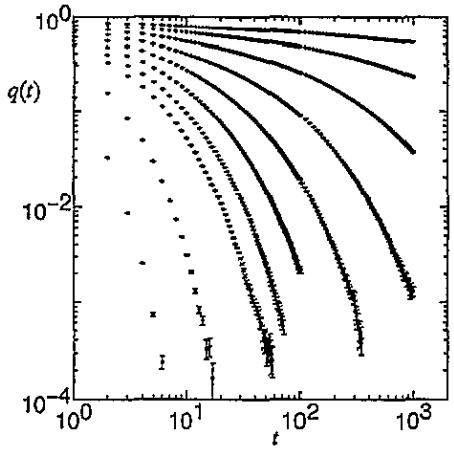

Figure 1. Autocorrelation function $q(t)$ for $T =$ 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, 2.0, 2.2, 3.0 and 5.0 (top to bottom).

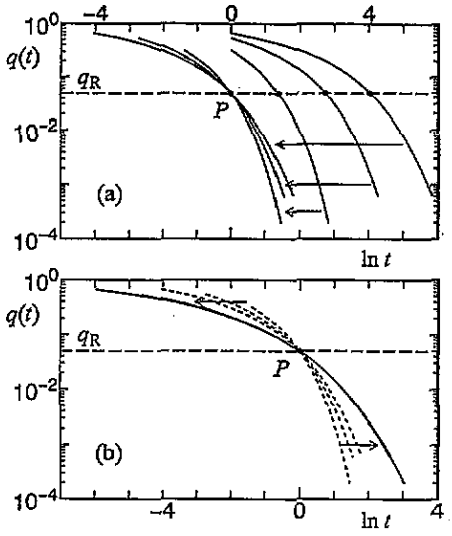

Figure 2. Schematic plot of q versus $\ln t$ for explanation of the shift-and-stretch method: (*a*) shift and (*b*) stretch; the three curves represent data for $T = 1.6$, 2.0 and 3.0. The reference value q_R is indicated by the dashed line (see text).

(i) Take a reference value q_R. We take $q_R = 0.05$ here, for convenience of explanation, as indicated by the dashed line in figure 2(*a*).

(ii) Shift each curve to the left so that the point crossing q_R coincides with $P = (0, q_R)$. The displacement corresponds to the *relative* value of $\ln \tau$.

(iii) Stretch the abscissa by multiplying the value of the slope at P for each curve (figure 2(*b*)). The scale of magnification is proportional to A.

After this procedure these curves tangentially cross each other at P, and if the scaling (3) holds they merge globally, as shown in figure 2(*b*), irrespective of the value of q_R. In the actual scaling the above 'shift-and-stretch' procedure was performed more systematically. For this purpose we generalized Mackinnon's method [9] in order to deal with the exponent A as well as τ (see appendix for details). The full result shown in figure 3 exhibits an excellent success of the scaling. The characteristic feature of the scaling (3) is that only $\ln t$ is scaled (abscissa) but the value of q itself (ordinate) is not modified. This is in contrast with the traditional dynamical scaling.

Although we cannot (and need not) determine the functional form of $Q(z)$ itself from the scaling, one may assume it to be $Q(z) = \exp(-z)$, and then $q(t)$ can be expressed as the stretched exponential law, $q(t) = \exp[-(t/\tau)^A]$. We tried to shift and stretch figure 3 so that $Q(z)$ looks like $\exp(-z)$ as much as possible. In figure 4 we show the result on a semi-log plot, where $\exp(-z)$ is indicated by the solid line. This assumption works well over the wide range of relaxation observed up to $q \simeq 10^{-3}$.

3.2. Scaling parameters $A(T)$ and $\tau(T)$

From the scaling analysis described above, we can obtain $A(T)$ and $\tau(T)$ except for the common multiplicative factors (see appendix). The ambiguity comes from

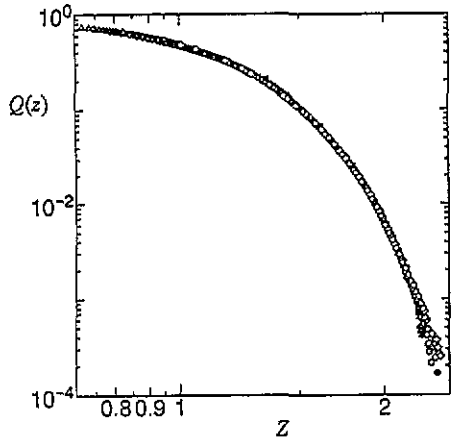

Figure 3. Scaled $q = Q(z)$ versus $z = (t/\tau)^A$. The data used are the same as in figure 1.

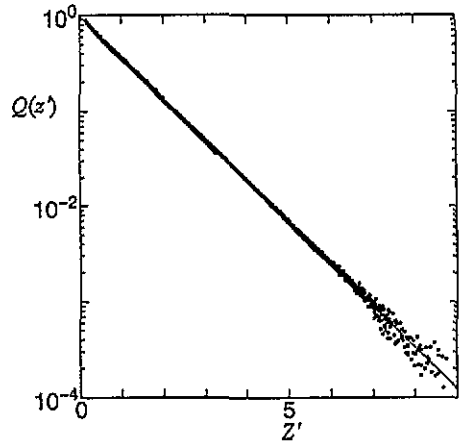

Figure 4. Rescaled $Q(z)$. The solid line indicates $Q(z) = \exp(-z)$.

the fact that if $Q(z)$ is a scaling function satisfying (3), $\tilde{Q}(\tilde{z}) = Q(a\tilde{z}^b)$ also satisfies the equation with the rescaled parameters $\tilde{A} = A/b$ and $\tilde{\tau} = a^{1/A}\tau$. This corresponds to global shift-and-stretch in figure 3. To fix these factors, we take the assumption mentioned in the previous subsection, that is, the functional form is to be $Q(z) = \exp(-z)$. These parameters so obtained to produce figure 4 are summarized in figures 5 and 6.

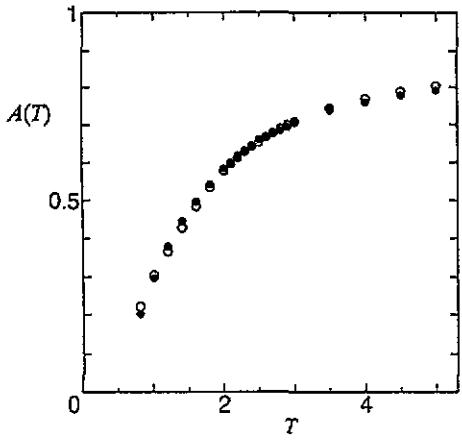

Figure 5. Temperature dependence of the scaling exponent $A(T)$ obtained from the scaling in figure 4 (full circles) and from direct fit to $q(t) = \exp[-(t/\tau)^A]$ (open circles).

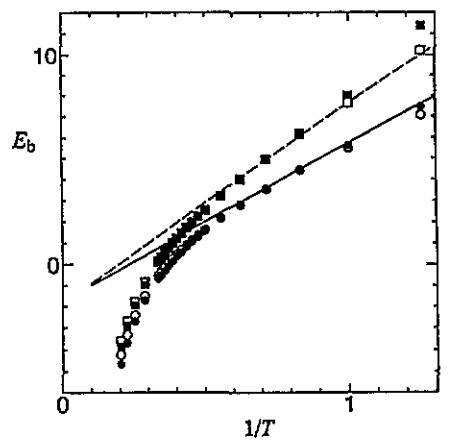

Figure 6. Energy barrier $E_b = T \ln \tau$ versus $1/T$ obtained from the scaling in figure 4 (full circles) and from direct fit to $q(t) = \exp[-(t/\tau)^A]$ (open circles). The asymptotic form is indicated by a solid line. $T \ln \tau_{av}$ evaluated by (5) is also shown by full squares (*our scaling*) and open squares (*direct fit*). The dashed line represents the result taken from [10].

As shown in figure 5 $A(T)$ has a cross-over temperature around T_c^{pure}, below

which it depends almost linearly on T. The result resembles that of β obtained by Ogielski for the 3D system [6] and this dependence is widely seen in the various systems which show the stretched exponential decay of relaxation. For comparison we directly fitted our data to the form $q(t) = \exp[-(t/\tau)^A]$ for each temperature. The exponent is also plotted in the figure, from which the validity of the assumption $Q(z) = \exp(-z)$ is confirmed.

The other parameter τ, representing the relaxation time, is shown in figure 6. Although we could not reach temperatures sufficiently low to determine whether the energy barrier, $E_b = T \ln \tau$, really diverges at $T = 0$, we found that the slope of $\ln(\ln \tau)$ against $\ln T$ seems to saturate around 2 rather than 1 for small T. Therefore we tentatively estimated its asymptotic behaviour as $E_b \simeq 7.47/T - 1.73$, which is indicated by the solid line in figure 6. The result of individual fitting to the stretched exponential form mentioned above is also shown by open circles.

This behaviour of τ is consistent with the result of Young [10, 11], who analysed the averaged correlation time

$$\tau_{av} = \int_0^\infty q(t)\,dt. \tag{4}$$

By substituting the stretched exponential form $q(t) = \exp[-(t/\tau)^A]$ to the above equation, we get the relation

$$\tau_{av} = \tau\Gamma(A^{-1})/A. \tag{5}$$

This means that, if $A \propto T$, the difference is $T \ln \tau_{av} - T \ln \tau \sim \ln T$ in very low temperatures, which is less divergent than T^{-1}. By using the relation (5) we evaluated $T \ln \tau_{av}$ from our result, which is shown by squares in figure 6. For comparison the result of Young taken from [10] is indicated by the dashed line. One can see the quantitative agreement of both results.

4. Comparison to other relaxation laws

4.1. Stretched exponential law

Ogielski fitted his result of $q(t)$ for the 3D Ising SG system to the form

$$q(t) \sim t^{-x} \exp[-(t/\tau)^\beta] \tag{6}$$

and obtained β almost proportional to T for $T_g < T < T_c^{pure}$ [6]. Below T_g the exponent could not be determined because τ diverges there and the relaxation obeys the power law.

There is an interesting correspondence between the function (6) and our scaling (3). Equation (6) is compatible with (3) if the exponent x is proportional to β. Our data, however, do not seem to have such a relation between these two exponents. Another possibility is $x = 0$ and then the assumption $Q(z) = \exp(-z)$ makes our scaling (3) consistent with (6). Note that the absence of t^{-x} in (6) implies the absence of of the critical slowing down, i.e. of the SG transition at a finite temperature where τ should diverge, and it is indeed the present case.

4.2. Enhanced power law

The cluster theory predicts the relaxation obeying the enhanced power law below T_c

$$q(t) \simeq \exp[-C(\ln t)^{d/(d-1)}] \tag{7}$$

where d denotes the space dimension of the lattice under consideration [1]. In figure 7 we plot $\ln(-\ln q)$ versus $\ln(\ln t)$, where the exponent $d/(d-1)$ (= 2 in the present case) is represented by the long-time limit of the slope. For high temperatures the exponent is, if it exists, obviously greater than 2 while the value for low temperatures cannot be determined from this plot. To see the tendency of the slope we take the derivative in figure 8. It seems that the slope tends to increase above 2 and does not saturate at any value, at least not within the time scale observed, which indicates that the relaxation is faster than (7).

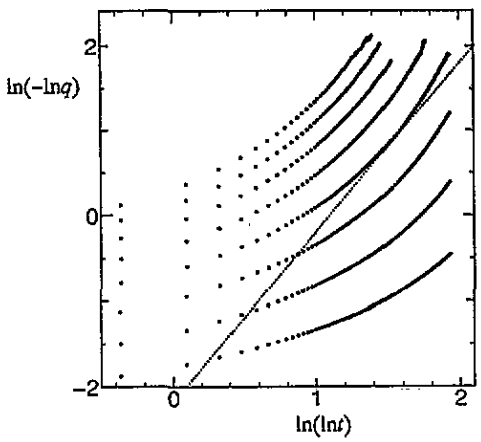

Figure 7. $\ln(-\ln q)$ versus $\ln(\ln t)$ for $T = 2.4$, 2.0, 1.8, 1.6, 1.4, 1.2, 1.0 and 0.8 (top to bottom). The expected slope $d/(d-1) = 2$ is indicated by the dashed line.

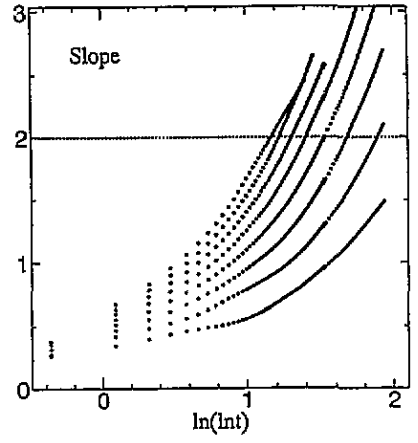

Figure 8. The first derivative of the curves in figure 7.

4.3. Logarithmic decay

Our scaling (3) can describe logarithmic decay for low temperatures. If A is small so that $Q(z)$ can be expanded around $z = 1$, equation (3) is approximated by

$$q(t) \simeq Q(1) - Q'(1)A(T)\ln(t/\tau). \tag{8}$$

Assuming $A(T) \propto T$ we obtain $T \ln t$ behaviour of relaxation.

The $T \ln t$ relaxation is often explained on the basis of the Néel model that is incorporated into the distribution of barriers $P(E)$ [12]. In this model each barrier E is associated with a single Debye relaxation having $\tau(E) = \tau_0 \exp(E/T)$ and the total relaxation is expressed as the superposition

$$q(t) = \int \mathrm{d}E\, P(E) \exp[-t/\tau(E)] \tag{9}$$

$$q(t) \simeq I[T \ln(t/\tau_0)] \tag{9'}$$

where $I(z) = \int_z \mathrm{d}E\, P(E)$ is the universal function. Equation (9′) looks similar to our scaling form (3) if $A \propto T$ although τ_0 is T independent. We emphasize, however, that the origins of T in $T \ln t$ are different. The exponent A represents the modification from the Debye relaxation and possibly characterizes the distribution function of E. Therefore the T dependence should be explained as a result of the T dependence of $P(E)$ [13].

5. Discussion

We do not insist that our scaling (3) be valid in all time regions. Generally speaking, it is very difficult to estimate the time scale τ_a where the final asymptotic region begins. It is generally different from the relaxation time τ and seems to have barely been achieved by simulations even at high temperatures. This is really the case for the one-dimensional Ising system, in which the exact solution for $q(t)$ can be obtained. For the Glauber dynamics it is expressed as $q(t) = \sum_n \eta^{|n|} \exp(-t) I_n(\gamma t)$, where $I_n(x)$ is the modified Bessel function of nth order and $\eta = \tanh(1/T)$, $\gamma = \tanh(2/T)$. From the expression one can see that the asymptotic region is $t \gg \tau_a = 1/\gamma$, where $q(t) \sim t^{-1/2} \exp(-t/\tau)$ with $\tau = (1-\gamma)^{-1}$. For the sequential spin-flip dynamics one can find that τ_a is almost T-independent while $\tau \propto 1/\ln T$ for large T. In any case $\tau \ll \tau_a$ in the high-temperature region. A similar situation may occur for the present system and we think that this is why $A(T)$ is still smaller than unity above T_c^{pure} in figure 5, where the final asymptotic form should be a simple exponential function. Therefore we cannot exclude the possibility that a scaling discrepancy is seen for smaller q.

Nevertheless the present result indicates that the scaling form (3) can describe, at the very least, the pre-asymptotic behaviour of $q(t)$ and this is what we observe on a realistic time scale. In fact there are many experimental studies on the remanent magnetization whose results show the stretched exponential relaxation (6) with very small x (less than 0.1 or so, see [14], for example). They are obtained from observations at the early stage relative to the relaxation time (very long in real time, however), where the remanent magnetization remains not less than 20%. This situation is very similar to that of our and many other simulations. What is controversial is that there exists a sharp transition temperature T_g and the slow relaxation seems to occur below T_g, whereas the relaxation behaves anomalously above T_g, if there is any, in simulation. One of the interpetations is that in real materials the apparent transition is a result of a dynamical effect and the variance of τ is much sharper than in the simulation.

Our scaling also includes logarithmic relaxation as discussed in section 4.3. We consider that the $T \ln t$ behaviour of the relaxation can be explained as a result of the smallness of the exponent A.

In conclusion, we have proposed a new scaling (3) for the autocorrelation function of the 2D ±J Ising SG. The scaling works well in the case $T_g = 0$ and at least up to the intermediate regime of relaxation, and gives a unified description of $q(t)$ over a wide range of temperatures.

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Appendix

We want to scale $q(t)$ for various temperatures to the following form

$$q(t;T) = Q[(t/\tau(T))^{A(T)}] = \tilde{Q}[A(T)\ln t - A(T)\ln\tau(T)] \tag{A1}$$

and extract the relaxation time $\tau(T)$ and the exponent $A(T)$. Since $q(t)$ is a monotonically decreasing function, (A1) can be expressed as

$$\tilde{Q}^{-1}(q) = A(T)\ln t - A(T)\ln\tau(T). \tag{A1'}$$

This form shows that the inverse function $t(q)$ can be scaled by a simple linear transformation.

Assume that we have observed data $q(t;T_i)$ for n different temperatures T_i $(i = 1,\ldots,n)$. For each i let the minimum and maximum of the data be q_i^0 and q_i^1, respectively, so that $q(t;T_i) \in I_i = [q_i^0, q_i^1]$. Now we define

$$y_i(q) = \ln t(q;T_i) \tag{A2}$$

$$X_i(q) = a_i y_i(q) + b_i \tag{A3}$$

where $t(q;T_i)$ is the inverse function of $q(t;T_i)$. If the scaling (A1) holds there exist a_i and b_i such that $X_i(q)$ is independent of i and then $a_i = cA(T_i)$ and $b_i = -cA(T_i)\ln\tau(T_i) + b_0$, where c and b_0 are constants.

To estimate the best values of $\{a_i\}$ and $\{b_i\}$, we consider making the difference between X_i and X_j for each i and j as small as possible. For this requirement one may minimize the following cost function

$$S_0(\{a_i\},\{b_i\}) = \tfrac{1}{2}\sum_{i,j}\langle (X_i - X_j)^2\rangle_{ij} \tag{A4}$$

where

$$\langle f\rangle_{ij} = \int_{I_{ij}} \mathrm{D}q\, f(q) \qquad I_{ij} = I_i \cap I_j \tag{A5}$$

(the measure $\mathrm{D}q$ was chosen to be $\mathrm{d}\ln q$ in our procedure). Equation (A4), however, has the trivial absolute minimum at $a_i = 0$ and $b_i = b_0$ for all i, at which $S_0 = 0$. To get rid of this triviality we impose a condition $\sum_i a_i^2 = 1$ on the minimization. Introducing a Lagrange multiplier λ, we take our new cost function to be

$$\begin{aligned} S(\{a_i\},\{b_i\}) &= S_0 - \lambda\Big(\sum_i a_i^2 - 1\Big) = \sum_{i,j}\left(\langle X_i^2\rangle_{ij} - \langle X_i X_j\rangle_{ij}\right) - \lambda\Big(\sum_i a_i^2 - 1\Big) \\ &= \sum_{i,j}\left(\langle y_i^2\rangle_{ij} a_i^2 + 2\langle y_i\rangle_{ij} a_i b_i + \langle 1\rangle_{ij} b_i^2\right) \\ &\quad - \sum_{i,j}\left(\langle y_i y_j\rangle_{ij} a_i a_j + 2\langle y_i\rangle_{ij} a_i b_j + \langle 1\rangle_{ij} b_i b_j\right) - \lambda\Big(\sum_i a_i^2 - 1\Big) \\ &= \boldsymbol{a}^{\mathrm{T}}\mathbf{C}_2\boldsymbol{a} + 2\boldsymbol{a}^{\mathrm{T}}\mathbf{C}_1\boldsymbol{b} + \boldsymbol{b}^{\mathrm{T}}\mathbf{C}_0\boldsymbol{b} - \lambda(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} - 1) \end{aligned} \tag{A6}$$

where

$$[\mathbf{C}_2]_{ij} = \delta_{ij} \sum_k \langle y_i^2 \rangle_{ik} - \langle y_i y_j \rangle_{ij}$$

$$[\mathbf{C}_1]_{ij} = \delta_{ij} \sum_k \langle y_i \rangle_{ik} - \langle y_i \rangle_{ij} \tag{A7}$$

$$[\mathbf{C}_0]_{ij} \delta_{ij} \sum_k \langle 1 \rangle_{ik} - \langle 1 \rangle_{ij} = \delta_{ij} \sum_k |I_{ik}| - |I_{ij}|.$$

Note that by virtue of the linearity of (A1′) or equivalently (A3), our cost function has a simple bilinear form and can easily be minimized. Hereafter we make two assumptions. One is that for all i there exists j such that $I_{ij} \neq 0$, which means that there are no isolated data. This assumption excludes the trivial singularity of the above matrices. The other is that $\mathbf{C}_0$ cannot be decomposed into submatrices. Otherwise the procedure below is applied for each submatrix independently.

Taking the derivatives of (A6), we get the following equations:

$$\mathbf{C}_2 \boldsymbol{a} + \mathbf{C}_1 \boldsymbol{b} - \lambda \boldsymbol{a} = \mathbf{0} \tag{A8}$$

$$\mathbf{C}_1^{\mathrm{T}} \boldsymbol{a} + \mathbf{C}_0 \boldsymbol{b} = \mathbf{0}. \tag{A9}$$

We want to express $\boldsymbol{b}$ in terms of $\boldsymbol{a}$ by using (A9) but unfortunately the inverse of $\mathbf{C}_0$ does not exist because the uniform vector $\boldsymbol{u}$, whose elements are all 1, is a right eigenvector of $\mathbf{C}_0$ with zero eigenvalue (this is also true for $\mathbf{C}_2$ and $\mathbf{C}_1$). This property is associated with the fact that for all i and j, $X_i - X_j$ are invariant under the uniform change of b_i to $b_i + b_0$. To get around this difficulty we restrict $\boldsymbol{b}$ so as to satisfy $\boldsymbol{u}^{\mathrm{T}} \boldsymbol{b} = 0$. Let $\boldsymbol{v}_\mu$ be the normalized eigenvector of $\mathbf{C}_0$ with the eigenvalue λ_μ. Then we can construct the pseudo-inverse of $\mathbf{C}_0$ as:

$$\mathbf{C}_0^{+} = \sum_{\mu, \lambda_\mu \neq 0} \lambda_\mu^{-1} \boldsymbol{v}_\mu \boldsymbol{v}_\mu^{\mathrm{T}}. \tag{A10}$$

If $\boldsymbol{u}$ is the only eigenvector of $\mathbf{C}_0$ whose eigenvalue is 0 then

$$\mathbf{C}_0^{+} \mathbf{C}_0 \boldsymbol{b} = \boldsymbol{b} \qquad \text{if } \boldsymbol{u}^{\mathrm{T}} \boldsymbol{b} = 0 \tag{A11}$$

and from (A9) we get

$$\boldsymbol{b} = -\mathbf{C}_0^{+} \mathbf{C}_1^{\mathrm{T}} \boldsymbol{a} \tag{A12}$$

which is substituted in (A8) to obtain the equation for $\boldsymbol{a}$

$$(\mathbf{C}_2 - \mathbf{C}_1 \mathbf{C}_0^{+} \mathbf{C}_1^{\mathrm{T}}) \boldsymbol{a} = \lambda \boldsymbol{a}. \tag{A13}$$

By using (A12) and (A13) we can evaluate the cost function (A6) as

$$S = \boldsymbol{a}^{\mathrm{T}} \mathbf{C}_2 \boldsymbol{a} - \boldsymbol{a}^{\mathrm{T}} \mathbf{C}_1 \mathbf{C}_0^{+} \mathbf{C}_1^{\mathrm{T}} \boldsymbol{a} - \lambda (\boldsymbol{a}^{\mathrm{T}} \boldsymbol{a} - 1) = \lambda. \tag{A14}$$

Thus $\boldsymbol{a}$ is found to be the eignevector associated with the smallest eigenvalue of (A13). Substituting it in (A12) we get the corresponding $\boldsymbol{b}$.

References

[1] Randeria M, Sethna J P and Palmer R G 1985 *Phys. Rev. Lett.* **54** 1321
[2] Griffiths R B 1969 *Phys. Rev. Lett.* **23** 17
[3] Jain S 1988 *J. Phys. C: Solid State Phys.* **21** L1045
[4] Colborne S G W and Bray A J 1989 *J. Phys. A: Math. Gen.* **22** 2505
[5] Takano H and Miyashita S 1989 *J. Phys. Soc. Japan* **58** 3871
[6] Ogielski A T 1985 *Phys. Rev.* B **32** 7384
[7] Nemoto K, Hukushima K and Takayama H 1992 *Computational Approaches in Condensed Matter Physics* ed M Imada, S Miyashita and H Takayama (Berlin: Springer) p 242
[8] Ito N, Taiji M and Suzuki M 1992 *Computational Approaches in Condensed Matter Physics* ed M Imada, S Miyashita and H Takayama (Berlin: Springer) p 287
[9] Olive J A, Young A P and Sherrington D 1986 *Phys. Rev.* B **34** 6341
[10] Young A P 1983 *Phys. Rev. Lett.* **50** 917
[11] Young A P 1984 *J. Phys. C: Solid State Phys.* **18** L517
[12] McMillan W L 1983 *Phys. Rev.* B **28** 5216
[13] Nemoto K and Takayama H 1983 *J. Phys. C: Solid State Phys.* **16** 6835
[14] Alba M, Ocio M and Hammann J 1986 *Europhys. Lett.* **2** 45